Model Categories – Simone $\boxed{\text{Talk 5}}$

Everything here is due to Quillen & extends homotopy theory to a more general context.

Def Let $\mathcal{C}$ be a category

$$K \subset \mathrm{Mor}(\mathcal{C})$$

f has the left lifting property (LLP) w.r.t. K if

we have
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "\in K"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists h, ?"] & B \end{tikzcd}$$

We say f is K-projective

f has the RLP w.r.t. K if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "K \ni"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed, "\exists h, ?"] & Y \end{tikzcd}$$

We then say ~~f~~ is K-injective.

Def A weak factorization system (WFS) consists of two subsets $\mathrm{Proj}, \mathrm{Inj} \subset \mathrm{Mor}(\mathcal{C})$

a) for $f \in \mathcal{C}(X,Y)$, $\exists\, i \in \mathrm{Inj}$, $p \in \mathrm{Proj}$ s.t. $f = i \circ p$

2) Proj is precisely the class of morphisms with LLP wrt Inj & vice versa.

Def A <u>category with weak equivalences</u> is a category $\mathcal{C}$ and $W \subseteq \mathrm{Mor}(\mathcal{C})$ s.t.

1) f isomorphism $\Rightarrow f \in W$

2) if f, g are composable & two of f, g, $f \circ g$ are in W, so is the third [2-out-of-3]

Def A <u>model category</u> is a category w/ w.e. $(\mathcal{C}, W)$ along with ① $\mathrm{Fib}, \mathrm{Cof} \subseteq \mathrm{Mor}(\mathcal{C})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ & $(\mathrm{Cof} \cap W, \mathrm{Fib})$ are weak fact. systems ~~and~~ ② $\mathcal{C}$ has all small (co) limits.

Ex the Quillen-Serre ~~model~~ model category on Top

W = weak homotopy equivalences

$f \in W$ means $\pi_0(f)$ is a bijection & $\pi_n(f)$ is a group isomorphism.

Fib = Serre fibrations

f has RLP w.r.t. to $\{ D^n \xhookrightarrow{(id,0)} D^n \times I \}$

Cof = retracts of relative cell complexes

Prop $\mathcal{C}$ a category & $K \subseteq \mathrm{Mor}(\mathcal{C})$

1. f iso $\Rightarrow$ $f \in K$-inj & $f \in K$-proj

2. K-inj is closed under composition, retracts, pullbacks in $\mathcal{C}$, & products in $\mathrm{Arr}(\mathcal{C})$

3. K-proj is closed under comp., retracts, pushouts in $\mathcal{C}$, coproducts in $\mathrm{Arr}(\mathcal{C})$, and transfinite composition.

Pf good exercise! $\square$

Prop (Retract argument)

Consider
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & A \arrow[ur, "p"'] & \end{tikzcd}$$

1) If f has LLP against p, then f is a retract ~~with~~ of i.

2) If f has RLP w.r.t. i, " " is a retract of p.

Pf
$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, "h"] & Y \end{tikzcd} \Rightarrow \begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"] & A \arrow[r, "p"] & Y \end{tikzcd}$$ is retract $\square$

Def Let $(\mathcal{C}, W)$ be a category with weak equivalences.

The localization at W is a category equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ s.t.

1) $f \in W \Rightarrow \gamma(f)$ is an iso

2) whenever a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(f)$ iso if $f \in W$, there is a factorization

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Downarrow\cong"] & \mathcal{D} \\
& \mathcal{C}[W^{-1}] \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

unique up to natural isomorphism.

Def An object X in a model category $\mathcal{C}$ is cofibrant if $0 \to X$ is a cofibration and fibrant if $X \to 1$ is a fibration.

Every object admits a cofibrant replacement

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr, "\mathrm{Cof} \ni"'] & & X \\
& QX \arrow[ur, "\in \mathrm{Fib} \cap W"'] &
\end{tikzcd}$$

and a fibrant replacement

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "\mathrm{Cof} \cap W \ni"'] & & 1 \\
& RX \arrow[ur, "\in \mathrm{Fib}"'] &
\end{tikzcd}$$

Def A <u>path object</u> of $X \in \mathcal{C}$ is a factorization

$$\begin{tikzcd} X \arrow[r, "i"', "\in W"] \arrow[rr, bend right, "\Delta_X = (id, id)"'] & \mathrm{Path}(X) \arrow[r, "(p_0, p_1)", "\in Fib"'] & X \times X \end{tikzcd}$$

A <u>cylinder object</u> is a factorization

$$\begin{tikzcd} X \sqcup X \arrow[rr, "id + id"] \arrow[dr, "i_0 + i_1", "\in Cof"'] & & X \\ & \mathrm{Cyl}(X) \arrow[ur, "\in W"'] & \end{tikzcd}$$

Def $f, g \in \mathcal{C}(X, Y)$

A <u>left homotopy</u> $\eta : f \underset{L}{\Rightarrow} g$ is a ~~cylinder map~~ diagram

$$\begin{tikzcd} X \sqcup X \arrow[rr, "f + g"] \arrow[dr, "i_0 + i_1"'] & & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\eta"'] & \end{tikzcd}$$

A <u>right homotopy</u> $\varepsilon : f \underset{R}{\Rightarrow} g$ is a diagram

$$\begin{tikzcd} X \arrow[rr, "(f, g)"] \arrow[dr, "\varepsilon"'] & & Y \times Y \\ & \mathrm{Path}(Y) \arrow[ur] & \end{tikzcd}$$

Prop $X, Y \in \mathcal{C}$ s.t. X is cofibrant & Y is fibrant

Then $\underset{L}{\Rightarrow}$ & $\underset{R}{\Rightarrow}$ agree & form an equivalence relation.

Thus $\mathcal{C}(X,Y)/\sim$

either notion of "htpy equivalence"

Let $\mathcal{C}_{cf}$ be the full subcategory of fibrant & cofibrant objects in $\mathcal{C}$.

Let $\mathcal{C}_{cf}/\sim$ denote the category with the quotient set of maps via htpy eq.

<u>Theorem</u> "Whitehead" For $\mathcal{C}$ a model category, If $f \in \mathcal{C}_{cf}(X,Y)$ & $f \in W$, then

$$f \simeq id$$

htpy eq

<u>Thm</u> For $\mathcal{C}$ a model category

$$Ho(\mathcal{C}) := \mathcal{C}[W^{-1}] \simeq \mathcal{C}_{cf}/\sim$$

(via the composite $\mathcal{C}_{cf} \hookrightarrow \mathcal{C} \longrightarrow Ho(\mathcal{C})$)

Def An adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

is a <u>Quillen adjunction</u> if F preserves cofibrations & acyclic cofibrations (dually U preserves fibrations & acyclic fibrations).

Ex

$$\begin{tikzcd} sSet \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & Top \arrow[l, bend left, "Sing"] \end{tikzcd}$$

$$\begin{tikzcd} Ch(R) \arrow[r, bend left, "Sym_R"] \arrow[r, phantom, "\perp"] & CAlg^{dg}(R) \arrow[l, bend left, "forget"] \end{tikzcd}$$

Def Consider a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \\ \mathcal{C} & \end{tikzcd}$$

<u>functors</u>

a <u>left Kan extension</u> of G along F

is a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[dl, "Lan_F G"] \\ \mathcal{C} & \end{tikzcd}$$

$\alpha : Lan_F G \circ F \Rightarrow G$

There is a <u>dual</u> notion of <u>right Kan extension</u>.

Def Let $\mathcal{C}$ be a model category

& $\gamma_{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ the localization.

Given $F: \mathcal{C} \longrightarrow \mathcal{D}$ a functor,

its <u>left derived functor</u> is

$$LF = \mathrm{Ran}_{\gamma_{\mathcal{C}}} F$$

its <u>right derived functor</u> is

$$RF = \mathrm{Lan}_{\gamma_{\mathcal{C}}} F$$

If $\mathcal{D}$ is a category with weak equivalences $W_{\mathcal{D}}$ given $\gamma_{\mathcal{D}}: \mathcal{D} \longrightarrow \mathcal{D}[W_{\mathcal{D}}^{-1}]$, then

the <u>total left derived functor</u> is

$$\mathbb{L}F = L(\gamma_{\mathcal{D}} \circ F)$$

its <u>total right derived functor</u> is

$$\mathbb{R}F = R(\gamma_{\mathcal{D}} \circ F)$$

Prop Given a Quillen adjunction $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ then we obtain an adjunction $\mathbb{L}F: \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}): \mathbb{R}F$.

Def A <u>Quillen equivalence</u> is a Quillen adjunction $F \dashv G$ s.t. $\mathbb{L}F \dashv \mathbb{R}G$ is an equivalence of categories.

Thm (Quillen)

Equip sSet with the model category structure where

$$W = \{ f : |f| \text{ is a weak htpy eq} \}$$

$$Cof = \{ f : \text{th } f_n : X_n \to Y_n \text{ is injective} \}$$

$$Fib = \{ f : f \text{ has RLP against } W \cap Cof \}.$$

Then

$$|\cdot| : sSet \rightleftarrows Top : Sing$$

is a Quillen equivalence.

Thm

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction & $\mathcal{C}$ a cofibrantly generated model category.

Let $f \in Mor(\mathcal{D})$ be a weak eq if $Uf \in W_{\mathcal{C}}$

$f \in Mor(\mathcal{D})$ be a fibration if $Uf \in Fib_{\mathcal{C}}$

Suppose ① U commutes with sequential colimits

② ~~if f has LLP w.r.t. all fibrations in D~~

if $f \in Mor\ \mathcal{D}$ is a cofibration & f has LLP w.r.t. all fibrations, then f is also a wk equiv in $\mathcal{D}$

(pins down acyclic cofib)

then $\mathcal{D}$ has a cofibrantly generated model structure with transferred generating sets